CS 245: Logic and Computation

Alice Gao

Lecture 2, September 12, 2017

Based on slides by Jonathan Buss, Lila Kari, Anna Lubiw and Steve Wolfman with thanks to B. Bonakdarpour, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek

Please come and sit in the front. I won't pick on you.

We will begin when the music stops.

Let's begin!

Announcements

Office hours

Regular office hours:

- Monday 11am-noon
- Tuesday 4-5pm

Email me to make an appointment if you want to see me at other times. Other times that I am often available are Monday 1-3pm and Wed 11am-noon.

Previously on CS 245

- The roadmaps of CS 245 (on the course website)
- What is logic?
- What are the applications of logic in computer science?
- What is a proposition?
- How do we translate English sentences into compound propositions?

A conjecture that is neither true nor false

Thank you, Christopher Lo, for this awesome story!

The Continuum Hypothesis by Georg Cantor: there is no set with cardinality strictly between the integers and the real numbers.

- Cantor failed to prove (or disprove) this conjecture.
- Kurt Godel proved that it cannot be disproved (1940).
- Paul Kohen proved that it cannot be proved (1963).
- Christopher's conclusion: there is no way to know for sure the truth value of this conjecture, or even that there is one.

https://cs.uwaterloo.ca/~alopez-o/math-faq/node71.html

http://www.salon.com/2013/07/14/how_does_one_prove_the_unprovable_partner/

Translating from English to Propositional Logic

Translate the following sentences to propositional logic formulas.

1. She is clever but not hard working.
2. I will eat an apple or an orange but not both.
3. If he does not study hard, then he will fail.
4. He will fail unless he studies hard.
5. He will not fail only if he studies hard.

English can be ambiguous.

Give multiple translations of the following sentences into propositional logic. Are these translations logically equivalent?

1. Pigs can fly and the grass is red or the sky is blue.
2. If it is sunny tomorrow, then I will play golf, provided that I do not feel stressed.

On an assignment, we may ask you to translate English sentences with ambiguity into propositional logic. I highly recommend that you explain the reasoning behind your solution.

Using propositional logic to model the real world

Check out the complete onnagata problem here: `https://www.student.cs.uwaterloo.ca/~cs245/Instructor-Specific-Pages/Alice_Gao/notes/PropLogic_onnagata_problem.pdf`.

Consider the following argument, drawn from an article by Julian Baggini. The onnagata are male actors portraying female characters in kabuki theatre.

Premise 1: If women are too close to femininity to portray women, then men must be too close to masculinity to play men, and vice versa.

Premise 2: And yet, if the onnagata are correct, women are too close to femininity to portray women and yet men are not too close to masculinity to play men.

Conclusion: Therefore, the onnagata are incorrect, and women are not too close to femininity to portray women.

Learning goals — revisited

By the end of the lecture, you should be able to

- Give a high-level description of logic.
- Give examples of applications of logic in computer science.
- Define propositions.
- Classify English sentences into propositions and non-propositions.
- Give multiple translations of English sentences with ambiguity.
- Translate English sentences with no ambiguity into compound propositions.

Propositional Logic: Syntax

Learning goals

By the end of this lecture, you should be able to:

- Describe the three types of symbols in propositional logic.
- Describe the recursive definition of well-formed formulas.
- Write the parse tree for a well-formed formula.
- Determine and give reasons for whether a given formula is well formed or not.
- Identify the recursive structure in a recursive definition.
- Explain how to use structural induction to prove properties of a recursively defined concept.

Atomic and compound propositions

An *atomic* proposition (also called an atom or an atomic formula) is a statement or an assertion that must be true or false. It is represented by a single propositional variable.

We construct a *compound* proposition by connecting atomic propositions using logical connectives.

Symbols and expressions

Propositions in English are represented by *formulas*.

A formula consists of a string of *symbols*.

There are three kinds of symbols.

Propositional variables: Lowercase Latin letters possibly with subscripts; e.g., p, q, r, p_1, p_2, q_{27}, etc.

Connectives: $\neg$, $\land$, $\lor$, $\rightarrow$ and $\leftrightarrow$.

Punctuation: '(' and ')'.

An *expression* is a string of symbols.

Examples of expressions:

- α: $(\neg)()\vee pq\rightarrow$
- β: $a \vee b \wedge c$
- γ: $((a \rightarrow b) \vee c)$

Expressions

An *expression* is a string of symbols.

Examples of expressions:

- α: $(\neg)()\vee pq\rightarrow$
- β: $a \vee b \wedge c$
- γ: $((a \rightarrow b) \vee c)$

What does each expression mean? In how many ways can we interpret each expression?

Expressions

An *expression* is a string of symbols.

Examples of expressions:

- α: $(\neg)()\vee pq\rightarrow$
- β: $a \vee b \wedge c$
- γ: $((a \rightarrow b) \vee c)$

What does each expression mean? In how many ways can we interpret each expression?

Ideally, we would like *one and only one way* to interpret each expression.

Can we focus on a set of expressions where each expression in this set has a unique interpretation?

Definition of well-formed formulas

Let $\mathcal{P}$ be a set of propositional variables. We define the set of *well-formed formulas* over $\mathcal{P}$ inductively as follows.

1. A single symbol of $\mathcal{P}$ is well-formed.
2. If α is well-formed, then $(\neg\alpha)$ is well-formed.
3. If α and β are well-formed, then each of $(\alpha \wedge \beta), (\alpha \vee \beta), (\alpha \rightarrow \beta), (\alpha \leftrightarrow \beta)$ is well-formed.
4. Nothing else is a well-formed formula.

Definition of well-formed formulas

Let $\mathcal{P}$ be a set of propositional variables. A *well-formed formula* over $\mathcal{P}$ has exactly one of the following forms:

1. A single symbol of $\mathcal{P}$,
2. $(\neg\alpha)$ if α is well-formed,
3. One of $(\alpha \wedge \beta), (\alpha \vee \beta), (\alpha \to \beta), (\alpha \leftrightarrow \beta)$ if α and β are well-formed.

Understanding well-formed formulas

Which symbol(s) can appear as the first one in a well-formed formula? Choose the largest set of possible symbols. (p denotes any propositional variable.)

a. $\neg$, $($

b. $\neg$

c. p, $\neg$, $($

d. p, $\neg$

e. p, $($

Let $\mathcal{P}$ be a set of propositional variables. A *well-formed formula* over $\mathcal{P}$ has exactly one of the following forms:

1. A single symbol of $\mathcal{P}$,
2. $(\neg\alpha)$ if α is well-formed,
3. One of $(\alpha \wedge \beta), (\alpha \vee \beta), (\alpha \rightarrow \beta), (\alpha \leftrightarrow \beta)$ if α and β are well-formed.

The parse tree of a well-formed formula

A parse tree is another way to represent a well-formed formula. The parse tree makes the structure of the formula explicit.

Write the parse tree of the following well-formed formulas:

1. $((a \lor b) \land (\neg(a \land b)))$
2. $(((\neg p) \land q) \rightarrow (p \land (q \lor (\neg r))))$.

Examples of well-formed formulas

Consider each formula below. Determine whether it's well-formed or not. If it is, write its parse tree. If it is not, explain why and describe ways to make it well-formed with minimum changes.

a. $\neg a$
b. $(a \to b)$
c. $(a \land b \land c)$
d. $(a \to b \to c)$
e. $(a \lor b \land c)$

(In case you are bored...) Challenge question: Alas, the instructors need your help coming up with exam questions on well-formed formulas. Create an exam question to test the concept of well-formed formulas. Please provide the question description, the correct answer, and explain what this question is testing. Email it to me (alice.gao@uwaterloo.ca) and I'll share the good ones with everyone.

Unique Readability of Formulas

Does every well-formed formula have a unique meaing? Yes.

Theorem. Every well-formed formula has a unique derivation as a well-formed formula. That is, each well-formed formula has exactly one of the following forms:

> (1) an atom, (2) $(\neg\alpha)$, (3) $(\alpha \wedge \beta)$, (4) $(\alpha \vee \beta)$, (5) $(\alpha \rightarrow \beta)$, or (6) $(\alpha \leftrightarrow \beta)$.

In each case, it is of that form in exactly one way.

(Why) Is the Theorem True?

As an example, consider $((p \wedge q) \to r)$. It can be formed from the two formulas $(p \wedge q)$ and r using the connective $\to$.

If we tried to form it using $\wedge$, the two parts would need to be "$(p$" and "$q) \to r$". But neither of those is a formula!

The statement holds for this example.

We will prove this theorem using structural induction (a type of mathematical induction).

How much do you remember about induction?

1. Why and when do we use mathematical induction to prove a theorem?
2. What are the two main steps in an induction proof?
3. In each step, what do you need to prove?

Induction over natural numbers

You may remember proving properties of natural numbers (0, 1, 2, 3, ...) using induction from MATH 135.

Let P be some property. We want to prove that every natural number has property P. That is, $P(0)$, $P(1)$, $P(2)$, $P(3)$, $P(4)$, ... , are all true.

Theorem: $P(k)$ is true where $k =$ 0, 1, 2, 3, ...

Induction over natural numbers

Theorem: $P(k)$ is true where $k =$ 0, 1, 2, 3, ...

Examples of induction proofs that you may remember:

a. Base step: Prove $P(0)$. Inductive step: Consider an arbitrary $k \geq 0$. Assume that $P(k)$ is true. Prove that $P(k+1)$ is true.
b. Base step: Prove $P(0)$. Inductive step: Consider an arbitrary $k \geq 0$. Assume that $P(0)$, $P(1)$, ..., $P(k)$ are true. Prove that $P(k+1)$ is true.
c. Base step: Prove $P(0)$ and $P(1)$. Inductive step: Consider an arbitrary $k \geq 0$. Assume that $P(k)$ is true. Prove that $P(k+2)$ is true.

An incorrect induction proof (If you have time, figure out why this proof does not work.)

- Base step: Prove $P(0)$. Inductive step: Consider an arbitrary $k \geq 0$. Assume that $P(k)$ is true. Prove that $P(k+2)$ is true.

Structural induction (1)

Step 1: Identify the recursive structure in the problem.

Theorem: Every well-formed formula has an equal number of opening and closing brackets.

Structural induction (1)

Step 1: Identify the recursive structure in the problem.

Theorem 1: Every *well-formed formula* has an equal number of opening and closing brackets.

Notes:

- "Well-formed formulas" are the recursive structures that we are dealing with.
- "Has an equal number of opening and closing brackets" is the property that we are going to prove that well formed formulas have. Some examples of other properties that we could prove: (1) contains at least one propositional variable, (2) has an even number of brackets, etc.

Structural induction (2)

Step 2: Identify each recursive appearance of the structure inside its definition. (A recursive structure is self-referential. Where in the definition of the object does the object reference itself?

Let $\mathcal{P}$ be a set of propositional variables. A *well-formed formula* over $\mathcal{P}$ has exactly one of the following forms:

1. A single symbol of $\mathcal{P}$,
2. $(\neg\alpha)$ if α is well-formed,
3. One of $(\alpha \wedge \beta), (\alpha \vee \beta), (\alpha \to \beta), (\alpha \leftrightarrow \beta)$ if α and β are well-formed.

Structural induction (2)

Step 2: Identify each recursive appearance of the structure inside its definition. (A recursive structure is self-referential. Where in the definition of the object does it the object reference itself?

There are 3 cases in the following definition. The definition references itself in cases 2 and 3, as highlighted below.

Let $\mathcal{P}$ be a set of propositional variables. A *well-formed formula* over $\mathcal{P}$ has exactly one of the following forms:

1. A single symbol of $\mathcal{P}$,
2. $(\neg\alpha)$ if *α is well-formed*,
3. One of $(\alpha \land \beta), (\alpha \lor \beta), (\alpha \to \beta), (\alpha \leftrightarrow \beta)$ if *α and β are well-formed*.

Structural induction (3)

Step 3: Divide the cases into: those without recursive appearances ("base cases") and those with ("inductive" cases).

Let $\mathcal{P}$ be a set of propositional variables. A *well-formed formula* over $\mathcal{P}$ has exactly one of the following forms:

1. A single symbol of $\mathcal{P}$,
2. $(\neg\alpha)$ if *α is well-formed*,
3. One of $(\alpha \wedge \beta), (\alpha \vee \beta), (\alpha \rightarrow \beta), (\alpha \leftrightarrow \beta)$ if *α and β are well-formed*.

Structural induction (3)

Step 3: Divide the cases into: those without recursive appearances ("base cases") and those with ("inductive" cases).

Case 1 has no recursive appearanrance of the definition. Cases 2 and 3 have recursive appearances of the definition.

Let $\mathcal{P}$ be a set of propositional variables. A *well-formed formula* over $\mathcal{P}$ has exactly one of the following forms:

1. A single symbol of $\mathcal{P}$, (base case)
2. $(\neg\alpha)$ if *α is well-formed*, (inductive case)
3. One of $(\alpha \wedge \beta), (\alpha \vee \beta), (\alpha \rightarrow \beta), (\alpha \leftrightarrow \beta)$ if *α and β are well-formed*. (inductive case)

A structural induction template

Problem: Prove that every recursive structure φ has property P.

Define $P(\varphi)$ to be the property in the problem.

Theorem: For every *recursive structure φ*, $P(\varphi)$ holds.

Proof by structural induction:

Base case: *For every base case you identified, prove that the recursive structure φ has property P.*

(Continued on the next slide)

A structural induction template

Induction step: *For each recursive case you identified, write an induction step*

Recursive case 1:

Induction hypothesis: Assume *each recursive appearance of the structure in this case* has property P.
Prove that the recursive structure φ has property P using the induction hypothesis.

Recursive case 2:

(State the induction hypothesis and use it to prove the theorem.)

(Possibly more recursive cases)

By the principle of structural induction, every recursive structure φ has property P. QED

CS 245: Logic and Computation

Alice Gao

Lecture 3, September 14, 2017

Based on slides by Jonathan Buss, Lila Kari, Anna Lubiw and Steve Wolfman with thanks to B. Bonakdarpour, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek

Please come and sit in the front. I won't pick on you.

We will begin when the music stops.

Let's begin!

Announcements

Previously on CS245

- Give multiple translations of English sentences with ambiguity.
- Describe the three types of symbols in propositional logic.
- Describe the recursive definition of well-formed formulas.
- Write the parse tree for a well-formed formula.
- Determine and give reasons for whether a given formula is well formed or not.
- A template for writing structural induction proofs.

Plan for today

- Review the definition of well-formed formulas.
- Analyze the recursive structure of the well-formed formula definition.
- Review the template for writing structural induction proofs.
- Prove the unique readability theorem using structural induction.

Proving the unique readability of well-formed formulas

Lemma 1: Every well-formed formula starts with a propositional variable or an opening bracket.

Lemma 2: Every well-formed formula has an equal number of opening and closing brackets.

Lemma 3: Every proper prefix of a well-formed formula has more opening brackets than closing brackets.

Theorem: There is a unique way to construct every well-formed formula.

Review question

Which of the following statements is incorrect?

a. In the base case of an induction proof, we prove the theorem directly without using any additional assumption.
b. In an inductive step of an induction proof, we prove the theorem by using the induction hypothesis (which assumes that the theorem is true for a simpler version of the recursive structure).
c. A well-formed formula is a recursive concept because of cases 2 and 3 in its definition.
d. To prove a property of a well-formed formula using structural induction, there is 1 base case and 2 inductive cases in the proof.
e. A well-formed formula could start with a propositional variable or a unary connective.

Proving the unique readability of well-formed formulas

Lemma 1: Every well-formed formula starts with a propositional variable or an opening bracket.

Proving the unique readability of well-formed formulas

Lemma 2: Every well-formed formula has an equal number of opening and closing brackets.

The induction step

Lemma 2: Every well-formed formula has an equal number of opening and closing brackets.

Consider the inductive step of our structural induction proof of this lemma. Consider the case in which we prove that if the well-formed formula is of the form $(\alpha \wedge \beta)$, then it has an equal number of opening and closing brackets. What is the induction hypothesis in this case?

a. α is a well-formed formula.

b. α has an equal number of opening and closing brackets.

c. Each of α and β is a well-formed formula.

d. Each of α and β has an equal number of opening and closing brackets.

e. $(\alpha \wedge \beta)$ has an equal number of opening and closing brackets.

The induction hypothesis

Lemma 2: Every well-formed formula has an equal number of opening and closing brackets.

Below is a fragment of a structural induction proof of this lemma. On which line did we use the induction hypothesis?

a. Suppose that φ is of the form $(\neg\alpha)$ where α is a well-formed formula. We want to prove that $(\neg\alpha)$ has an equal number of opening and closing brackets.

b. Let $op(x)$ and $cl(x)$ denote the number of opening and closing brackets of x respectively.

c. α is well-formed, so $op(\alpha) = cl(\alpha)$.

d. By the form of φ, we know that $op((\neg\alpha)) = 1 + op(\alpha)$ and $1 + cl(\alpha) = cl((\neg\alpha))$.

e. Therefore, $op((\neg\alpha)) = 1 + op(\alpha) = 1 + cl(\alpha) = cl((\neg\alpha))$.

Unbalanced brackets in a proper prefix of a formula

Lemma 3: Every proper prefix of a well-formed formula has more opening brackets than closing brackets.

A *proper prefix* of φ is a non-empty segment of φ starting from the first symbol of φ and ending before the last symbol of φ.

How many proper prefixes does a formula have?

A *proper prefix* of a formula φ is a non-empty segment of φ starting from the first symbol of φ and ending before the last symbol of φ.

Let α and β be well-formed formulas. Let a (and b) denote any proper prefix of α (and β). Using α, β, a, b, and $\wedge$, how many different proper prefixes of $(\alpha \wedge \beta)$ are there? Here are three examples: $($, $(\alpha$, and $(\alpha \wedge$.

a. 3
b. 4
c. 5
d. 6
e. 7

Unbalanced brackets in a proper prefix of a formula

Lemma 3: Every proper prefix of a well-formed formula has more opening brackets than closing brackets.

Unbalanced brackets in a proper prefix of a formula

Lemma 3: Every proper prefix of a well-formed formula has more opening brackets than closing brackets.

Complete the induction step. If you prefer writing, then write it down. If you prefer talking, then describe it to your neighbour.

Proving the unique readability theorem

Theorem: There is a unique way to construct every well-formed formula.

(See the handout for an outline of the proof.)

If you are done with the proof, think about this question: As a human, we can determine whether a formula is well-formed or not by looking at it. How does the computer do this? Could you design an algorithm which reads a formula and determines whether it's well-formed or not?